An Improved IRA Algorithm and Its Application in Critical Eigenvalues Searching for Low Frequency Oscillation Analysis

Chongru Liu, *Senior Member, IEEE*, Xiao Li, Pengfei Tian, and Mu Wang

***Abstract*—On the basis of implicitly restarted Arnoldi (IRA) method, an improved algorithm is proposed, in which the dimension of Krylov subspace is dynamically increased to compute eigenvalues in specified circle. First, the radius of searching circles is dynamically expanded through automatically increasing the number of eigenvalues and the dimension of Krylov subspace, based on the locking mechanism. Second, the region where the low frequency oscillation modes located is divided into small independent computing units, which are covered by specified searching circles. Third, the independent computing units can be calculated simultaneously with no effects on each other. The proposed method can avoid eigenvalues missing caused by inappropriate resetting of search number in equidistant-searching IRA method. Furthermore, no manual intervention is needed in the proposed method. Two systems with 570 and 5272 state variables are tested in this paper, and the results indicate that the proposed method is efficient, reliable, and practical.**

***Index Terms*—Critical eigenvalues, dynamically increasing the dimension, low frequency oscillation, implicitly restarted Arnoldi method, searching circle.**

I. Introduction

THE poorly-damped low frequency oscillation (LFO) has been found to be a world-wide problem in interconnected power systems [1],[2], which is of special interest for small signal stability analysis (SSSA). Computing the critical eigenvalues is essential for understanding the effects of control parameters on selected oscillatory modes and facilitating the design of control system [3]–[5].

With the ever-increasing scale of modern interconnected power system, the algorithms, such as QR algorithm, to calculate all the eigenvalues of a matrix have some main disadvantages: 1) They cannot take the advantages of the sparse matrix [6]; 2) For a large power system with numerous state variables, the calculation time is too long to be accepted and most of the eigenvalues are unwanted. For modern large scale power system, it is usually required to calculate only a specific subset of eigenvalues with certain features of interest [7]–[9]. For example, only a narrow triangle region, in which the damping ratio ranges from 0 to 0.05 and the frequency ranges from 0.1 to 2.0 Hz [1], [10] are concerned in LFO problems. Therefore, the partial eigenvalues calculation method is a feasible and effective approach for the identification of LFO modes in SSSA program.

Significant efforts have been expended to study and develop partial Eigen analysis methodologies [11]–[22]. Analysis of essentially spontaneous oscillations in power systems (AESOPS) algorithm [11], [12] is utilized to calculate the eigenvalues associated with the rotor angle modes by using a frequency response approach, in which only the rotor angle modes are considered [13]. The selective modal analysis approach [14], [15] is utilized to compute the concerned eigenvalues by constructing a reduced order model that involves variables relevant to the selected modes. It is incapable for very large scale power system [16]. The sequential methods [17], such as the power iterations, the Rayleigh quotient iterations and the Newton method, have been applied to compute critical eigenvalues based on single vector iterations. However, these methods either converge very slowly or require strict initialization parameters. The subspace methods such as the simultaneous method [17] and the Arnoldi method [3], [18]–[21] are commonly adopted for the computation of selective eigenvalues of large matrices. The computational efficiency of subspace methods can be increased by using the locking mechanism [8], [14].

Among all the partial eigenvalues methods, the Arnoldi method is believed to be the most efficient approach [19]–[21] as it improves the slow convergence in subspace iteration and also overcomes the instability that Lanczos may encounter. Typically, the implicitly restarted Arnoldi (IRA) method is usually applied combining with the shift-invert transformation [20], [22].

The IRA method has been successfully implemented in many SSSA tools, such as, PEALS [13], Small Signal Analysis Tool (SSAT) of DSA Tool [23], PSAT [24], and PSASP [25]. In most of them, the eigenvalues associated with LFO modes is scanned over a frequency range on the complex plane by a number of pre-set shift points. However, without knowing any distribution

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information about the desired eigenvalues in the target region, there is no reference to set the parameters especially for the shift points and the number of eigenvalues for each shift point. Improper parameters will result in extra computation or missing oscillation modes. More experiences about the system or repeated and manual adjustments are needed for the appropriate parameters setting. The essential reason of this problem is that all the eigenvalues in a fixed target region are required while the searching area cannot be given or limited in the existing algorithm.

Aiming at the above problem, an improved IRA algorithm is proposed in this paper. First, by increasing the dimension of Krylov subspace, the controllable parameter is transformed from the number of eigenvalues to the radius of searching circles. Therefore, in the proposed algorithm, the searching area can be controlled or given by the user. Second, the region to search, where the LFO modes located, is divided into small independent computing units, which are covered by the mentioned searching circles. These independent computing units can be calculated simultaneously in parallel, instead of in series, to reduce the calculation time. Third, the proposed method could avoid the missing of the eigenvalues calculated by inappropriate re-setting of search number.

The remaining parts of this paper are organized as follows. Section II covers the background and demonstrates existing problems. The proposed improved IRA algorithm and its implementation for the identification of LFO modes is detailed in Section III. In Section IV, the proposed method is verified via two actual systems with 570 and 5272 state variables. The conclusions are presented in Section V.

II. Basic Knowledge

A. The Mathematical Model of Power System

The mathematical model of the power system small-signal stability analysis can be formulated as a set of linearized differential equations as (1),

$$\begin{bmatrix} \Delta\dot{x} \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \tag{1}$$

where $\Delta x \epsilon R^n$ is the vector of state variables and $\Delta y \epsilon R^n$ is the vector of non-state variant. $\tilde{A} \in R^{n\times n}$, $\tilde{B} \in R^{n\times n}$, $\tilde{C} \in R^{n\times n}$, and $\tilde{D} \in R^{n\times n}$ are sparse matrixes which depend on the system parameters and operating point. Eliminating Δy in (1),

$$\Delta\, \dot{x} = A^+\, \Delta x \tag{2}$$

where $A^+ = \tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C}$. Although the state-space matrix A^+ is not a sparse matrix, all the calculation involving A^+ can be sparsely implemented through augmented system [17] as (1). The eigenvalues of A^+ are used to reveal the quantitative information of different stability modes for power system SSSA.

B. Implicitly Restarted Arnoldi Method [26]

The *m*-size Arnoldi factorization of a genetic matric *A* is shown in (3),

$$AV_m = V_m H_{\mathrm{m}} + f_m {e_m}^T \tag{3}$$

where $V_m \epsilon C^{n\times m}$. v_0 is the first column of V_m, and columns of V_m are a group of orthogonal basis of order-*m* Krylov subspace. Krylov subspace is the linear subspace spanned by the images of v_0 under the first m-1 powers of *A*. There is $k_m\,(A,\, v_0) = \mathrm{Span}\,\{v_0,\, Av_0,\, A^2 v_{0,}\, \ldots,\, A^{m-1} v_0\}$. $H_m \epsilon C^{m\times m}$ and $H_m = V_m^T A V_m$, is an upper Hessenberg matrix. $e_m \epsilon C^{m\times 1}$ is *m*-dimensional orthonormal basis whose last element is 1.

The IRA method combines the implicitly shifted QR scheme with *m*-step Arnoldi factorization. Implicitly restarting provides a means to extract the interesting information from large Krylov subspaces and continually compress the interesting information into a fixed size *k*-dimensional subspace. The IRA method includes four steps as follows [26].

Step 1: Input the initial vector v_0 and form the *k*-step Arnoldi factorization of *A* as (4),

$$AV_k = V_k H_k + f_k {e_k}^T \tag{4}$$

Step 2: Expand *k*-step to $m = k+p$ step.

$$AV_m = V_m H_{\mathrm{m}} + f_m {e_m}^T \tag{5}$$

Step 3: Find eigenpairs $(\lambda_i,\, h_i)$ of H_m and select *k* desired eigenpairs. The residual norm is

$$\|er_i\| = \|A\varphi_i - \lambda_i\varphi_i\| \;= \|f_m\| \;\left|e_m^T h_i\right| \tag{6}$$

If $\|er_i\| < \varepsilon\,(i = 1, 2 \ldots, k)$, the k desired eigenpairs $(\lambda_i,\, \varphi_i)$ of *A* can be obtained by $\varphi_i = V_m \cdot h_i$. If not, go to step 4.

Step 4: Select remaining $p\,(p = m - k)$ undesired eigenvalues of H_m as shifts for QR process, denoted by $\sigma_i\,(i = 1, 2, \ldots, p)$. By right multiplying Q to both sides of (5), it results in (7),

$$AV_m^+ = V_m^+ H_m^+ + f_m {e_m}^T Q \tag{7}$$

where $V_m^+ = V_m Q$, $H_m^+ = Q^T H_m Q$ and $Q = Q^{(1)}Q^{(2)} \cdots Q^{(p)}$. $Q^{(i)}$ is the orthogonal matrix of shifted QR algorithm. For each $Q^{(i)}$, there is $H_m - \sigma_i \cdot I = Q^{(i)} R^{(i)}$ and then update H_m to $[Q^{(i)}]^T H_m Q^{(i)}$. The first *k* columns on both sides of (7) provides an updated *k*-th step Arnoldi factorization,

$$AV_k^+ = V_k^+ H_k^+ + f_k^+ {e_k}^T \tag{8}$$

Replace (4) with (8) and go to step 2 until $\|er_i\| < \varepsilon\,(i = 1, 2 \ldots, k)$.

C. Shift Invert Transformation

For the SSSA of power systems, the critical eigenvalues of power systems are of special interest. However, these concerned critical eigenvalues are usually much smaller in modulus while the IRA method can easily converge to dominant eigenvalues with largest module. Therefore, the combination of the spectrum transformation and IRA method is usually used to calculate the critical eigenvalues. There are several spectrum transformation techniques, typically shift-invert transformation and Cayley transformation. The Shift-invert transformation uses a shift-point *u* and defines,

$$A \;= f_u \;\; \left(A^+\right) = \left(A^+ - uI\right)^{-1} \tag{9}$$

where *I* is the identity matrix. The eigenvalues of A^+ in the vicinity of *u* is mapped to the dominant eigenvalues of *A*. The eigenvalues of A^+ denoted by θ_i can be obtained by (10),

$$\theta_i = \frac{1}{\lambda_i} + u \tag{10}$$

D. Existing Problems

With the shifted-invert transformation, the controllable parameters of the IRA method are shift-point denoted by *u* and the number of eigenvalues closest to *u* denoted by *k*. Once the values of *u* and *k* are given, all the eigenvalues in a circle can be acquired. This circle's center is *u* and its radius is determined by the distribution density of eigenvalues near *u*.

Actually, the LFO analysis needs all the eigenvalues in a certain area, typically a narrow triangle region on the complex plane. However, there is no reference to set the value of *u* and *k* without knowing any eigenvalues distribution information. Therefore, it's one of the most urgent problems to transform controllable parameters from the number of eigenvalues to the searching range when using the IRA method for LFO modes identification. Furthermore, the online monitor-control and early-warning tools for power system are required to accomplish the SSSA calculation rapidly and automatically in searching all the critical eigenvalues in the concerned area without manual intervention.

III. Improved IRA Algorithm

A. Restart by Increasing the Dimension of Krylov Subspace

The *m*-step Arnoldi factorization can be noted in partitioned matrix form as (11),

$$A \begin{bmatrix} V_{nk} & V_{np} \end{bmatrix} = \begin{bmatrix} V_{nk} & V_{np} \end{bmatrix} \begin{bmatrix} H_{kk} & H_{kp} \\ H_{pk} & H_{pp} \end{bmatrix} + f_m e_m^T \tag{11}$$

Because of the Hessenberg form of H_m, only the element at 1st row one and *k*-th column is nonzero. It means that the elements of *k*-1 columns of $V_{np}H_{pk}$ are zero and $V_{np}H_{pk}$ would be $f_k e_k^T$. Thus, AV_{nk} will be,

$$A\, V_{nk} = V_{nk}\, H_{kk} + V_{np}\, H_{pk} = V_{nk}\, H_{kk} + f_k e_k^T \tag{12}$$

When $\|f_k\|$ is small enough to assume $f_k e_k^T$ and H_{pk} are almost zero, this process is called *deflation*. There will be,

$$A \begin{bmatrix} V_{nk} & V_{np} \end{bmatrix} = \begin{bmatrix} V_{nk} & V_{np} \end{bmatrix} \begin{bmatrix} H_{kk} & H_{kp} \\ 0_{pk} & H_{pp} \end{bmatrix} + \begin{bmatrix} 0_{nk} & f_m e_p^T \end{bmatrix} \tag{13}$$

Based on the convergence of *k* eigenvalues, *k* is expanded to $k + \Delta k$ and *m* is expanded to $m + \Delta m$. Denoting that $p' = (m + \Delta m) - (k + \Delta k)$, there is,

$$A \begin{bmatrix} V_{nk} & V_{n\Delta k} & V_{np'} \end{bmatrix} = \begin{bmatrix} V_{nk} & V_{n\Delta k} & V_{np'} \end{bmatrix} \begin{bmatrix} H_{kk} & H_{k\Delta k} & H_{kp'} \\ 0_{\Delta k k} & H_{\Delta k \Delta k} & H_{\Delta k p'} \\ 0_{p'k} & H_{p'\Delta k} & H_{p'p'} \end{bmatrix} + \begin{bmatrix} 0_{nk} & f_{\Delta k + p'} e_{\Delta k + p'}^T \end{bmatrix} \tag{14}$$

In the restart calculation of increased Δk eigenvalues, the interesting information will be compressed from $(m + \Delta m)$-dimensional subspace to $(k + \Delta k)$-dimensional subspace.

Otherwise, because the first *k* eigenvalues have been already obtained, the computational efficiency can be increased by locking the eigenvalues after they have converged [8], [14]. This implies that no more calculation is needed for the converged vectors until the termination of the algorithm. Therefore, during the increasing of subspace dimension, the searching range is sequentially enlarged for avoiding unnecessary computation based on the converged information.

B. The Transformation From Numbers to Searching Range

The *k* eigenvalues nearest to u, λ_i $(i = 1, 2 \cdots k)$, can be obtained in a single run of the IRA method combined with shifted-invert transformation. In the single run, the searching range is a circle, whose center is *u* and the radius is $\max\{|u - \lambda_i|\}$. Obviously, the radius of searching circle will increase with the extending of the dimension of Krylov subspace. If all the eigenvalues in a circle with specified radius *r* are desired, it can be implemented via following steps. First, a relatively small initial value of *k* should be given to map a small searching radius. Second, the value of *k* should be dynamically increased and the dimension of Krylov subspace should be dynamically expanded, until the searching radius reaches *r*. By doing this, the controllable parameter is transformed from the number of eigenvalues nearest *u* to the radius of searching circle with *u* as the center.

Without knowing any distribution information about the desired eigenvalues, it can avoid excessive calculation that the radius of searching circle is gradually expanded to *r* by dynamically increasing the dimension of subspace. This mechanism solves the problem caused by improper parameters and makes the searching process automatic without manual intervention. Furthermore, the calculation of increased Δk eigenvalues is based on the locking mechanism, which results in continuous and effective searching during the expansion of searching circle.

C. Locking Within an IRA-Iteration

After each restarting, $\|f_k\|$ is always almost of the same order of magnitude as $\max\{\|er_i\| : i = 1, 2 \ldots, k\}$. When $\max\{\|er_i\| : i = 1, 2 \ldots, k\} < \varepsilon$, there always is $\|f_k\| < \varepsilon$. Therefore, deflation and expanding dimension can be done at the same time. If deflation has finished, the column vectors in V_{nk} are considered locked. Therefore, the converged eigenvalues and associated eigenvectors can be decoupled from the active part of the iteration. In the next iteration, based on the convergence of *k* eigenvalues, expand *k* to *k*+Δk and *m* to *m*+Δm.

To clearly describe the proposed locking mechanism, it is supposed that there are *s* converged eigenvalues which have been locked, and *k* desired eigenvalues and *m*-dimension subspace (s<k<m<<n).

The *m*-step Arnoldi factorization of *A* becomes,

$$A \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} H_1 & H_3 \\ 0 & H_2 \end{bmatrix} + \begin{bmatrix} 0_{n,s} & f_m e_{m-s}^T \end{bmatrix} \tag{15}$$

where $V_1 \epsilon C^{n\times s}$, $V_2 \epsilon C^{n\times(m-s)}$, $H_1 \epsilon C^{s\times s}$, $H_2 \epsilon C^{(m-s)\times(m-s)}$. V_1 is the locked part and the subsequent implicit restarting should be done only on the basis V_2 and H_2.

Because $\lambda(H_m) = \lambda(H_1) \cup \lambda(H_2)$ and the eigenvalues of H_1 have obtained before, H_1 and H_3 need not to participate in the calculation of $\lambda(H_m)$. Actually, H_1 and H_3 are not required throughout the iterative process. All the eigenvalues of H_m can be obtained by only calculating the eigenvalues of H_2. Also, when selecting p ($p = m\text{-}k$) undesired eigenvalues from the eigenvalues of H_m as shifts for QR process, only the shifts QR factorization of H_2 needs to be calculated.

When updating V_m and H_m to V_k^+ and H_k^+ respectively in (8), we can just update V_2 and H_2.

According to (8), it results in (16) if Q is multiplied to both sides of (15).

$$A\,[V_1^+ \;\; V_2^+] = [V_1^+ \;\; V_2^+] \begin{bmatrix} H_1^+ & H_3^+ \\ 0 & H_2^+ \end{bmatrix} + [0_{n,s} \;\; f_m e_{m-s}^T Q_2] \tag{16}$$

where $V_1^+ = V_1 \cdot Q_1$, $V_2^+ = V_2 \cdot Q_2$, $H_1^+ = Q_1^T \cdot H_1 \cdot Q_1$, $H_2^+ = Q_2^T \cdot H_2 \cdot Q_2$, and $H_3^+ = Q_1^T \cdot H_3 \cdot Q_2$. Q_2 is equal to $Q_2^{(1)} Q_2^{(2)} \ldots Q_2^{(p)}$. For each $Q_2^{(i)}$, there is $H_2 - \sigma_i \cdot I = Q_2^{(i)} R_2^{(i)}$ and then update H_2 to $[Q_2^{(i)}]^T H_2 Q_2^{(i)}$.

H_1^+ and H_3^+ are not required to be calculated because H_1 and H_3 are not required in the next iteration, and V_1 does not need to update to V_1^+. Then, (16) can be expressed as (17),

$$A\,[V_1 \;\; V_2^+] = [V_1 \;\; V_2^+] \begin{bmatrix} H_1 & W_3^+ \\ 0 & H_2^+ \end{bmatrix} + [0_{n,s} \;\; f_m e_{m-s}^T Q_2] \tag{17}$$

where $W_3^+ = H_3 \cdot Q_2$. Comparing (16) and (17), H_2^+ is not affected even if V_1 is not updated to V_1^+. This can also be explained by the same subspace expanded by the column vector of V_1 and V_1^+. Since H_1 and W_3^+ can be discarded, just V_2^+ and H_2^+ should be calculated.

Therefore, the first k columns of (17) equal to the first k columns of (8). According to the analysis above, f_k^+ in (8) can be express only by H_2, Q_2 and V_2 as shown in (18).

$$f_k^+ = \beta_{k+1} \upsilon_{k+1} + \zeta_{k+1} f_m \tag{18}$$

where β_{k+1} is the entry in the (k + *1-s*)-th row and (*k-s*)-th column of $Q_2^T H_2 Q_2$, υ_{k+1} is the (k + *1-s*)-th column of $V_2 Q_2$, and ζ_{k+1} is the entry in the (k + *1-s*)-th row and (*k-s*)-th column of Q_2.

$\|er\|$ can be expressed only by f_m and H_2^+ which are irrelevant to H_1. Suppose (λ, $h = [h_1^T, h_2^T]^T$) is an eigenpairs of H_m, then,

$$\begin{bmatrix} H_1 & W_3^+ \\ 0 & H_2^+ \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} H_1 h_1 + W_3^+ h_2 \\ H_2^+ h_2 \end{bmatrix} = \lambda \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \tag{19}$$

$$\|er\| = \|f_m\| \;\; \left|e_m^T h\right| = \|f_m\| \;\; \left|e_{m-s}^T h_2\right| \tag{20}$$

According to (20), it can be seen that $\|er\|$ can be expressed by h_2, which is the eigenvector of H_2^+.

Consequently, the converged eigenvalues and associated eigenvectors can be completely decoupled in IRA calculation process. During the iteration process that extends the Arnoldi factorization from *k-step* to *m-step* ($m = k + p$), all of the columns of $[V_1, V_2]$ participate just as if no deflation had occurred. This assures that all of the new Arnoldi basis vectors are orthogonal to those converged Arnoldi basis vectors and the proposed locking technique does not affect the convergence of the algorithm. Furthermore, the proposed locking technique not only reduces the storage space needed but also saves computation time.

D. Automatic Given Value to Δk

If the value of Δk is small, the number of restart times will increase. If the value of Δk is large, the radius of searching circle might become larger than *r*, which leads to extra calculation.

To reduce calculation, the eigenvalues of H_m are utilized to obtain the proper value of Δk in this paper. When *k* eigenvalues of *A* have converged and the *k* eigenvalues could not cover the desired searching area, *k* has to be expanded to $k + \Delta k$ based on the *k* converged eigenvalues. Though the rest *p* ($p = m\text{-}k$) eigenvalues of H_m at this step have not converged, they are sort of estimation of the eigenvalues of *A* and can provide the approximate location of the rest eigenvalues of *A*.

If (λ, φ_m) is an approximation of eigenpairs (λ^*, φ_m^*) of *A* and $|\lambda - \lambda^*| \le \xi$ and $|\lambda - u| + \xi \le r$, it is assumed that the eigenvalue λ^* is within the desired range and λ^* is a desired eigenvalue. Suppose *q* is the number of specific eigenvalues satisfying the above conditions in the *p* estimated eigenvalues. The value of Δk is set as $q + 2$ to considering certain margin and ensure enough coverage of the searching circle.

If (λ_i, φ_i) is an approximation of eigenpairs (λ_i^*, φ_i^*) of *A*, there is,

$$\min_{\lambda_i^* \lambda(A)} |\lambda_i - \lambda_i^*| \le \kappa \|er_i\|_2 / \|\varphi_i\|_2 \tag{21}$$

where κ is the spectral condition number of *A* and $\kappa \ge 1$.

According to (21), the boundary ξ for the error of the estimated eigenvalues can be set as $\kappa\|er\|/\|\varphi_i\|$ or less than $\kappa\|er\|/\|\varphi_i\|$. Because V_m is unitary matrix and φ_i is orthonormal basis, $\|\varphi_i\|$ ($\varphi_i = V_m h_i$) equals 1. Hence, $\kappa\|er\|/\|\varphi_i\|$ is equal to $\kappa\|er\|$. Because $\kappa \ge 1$, there is $\kappa\|er\| \ge \|er\|$. In this paper, the boundary ξ is determined as $\|er\|$, which is less than $\kappa\|er\|/\|\varphi_i\|$ and easy to be obtained. When λ_i satisfies $|\lambda_i - u| + \|er_i\| \le r$, the residual norm $\|er_i\|$ is small and the estimated position of the eigenvalues of A is trustworthy.

Consequently, according to the H_m before next dimension extension, the Δk can be automatically set to $q + 2$ instead of a constant value. In this way, the value of Δk is relatively larger at the beginning of iterations, and the dimension extension is fast; the value of Δk is relatively smaller later, and the number of unwanted eigenvalues (eigenvalues outside the search range) will be less.

E. Algorithm Implementation

In order to ensure that the convergent eigenvalues can be locked, the condition of expanding dimension is limited to meet $\|er\| < \varepsilon$ and $\|f\| < \varepsilon$. The implementation process of

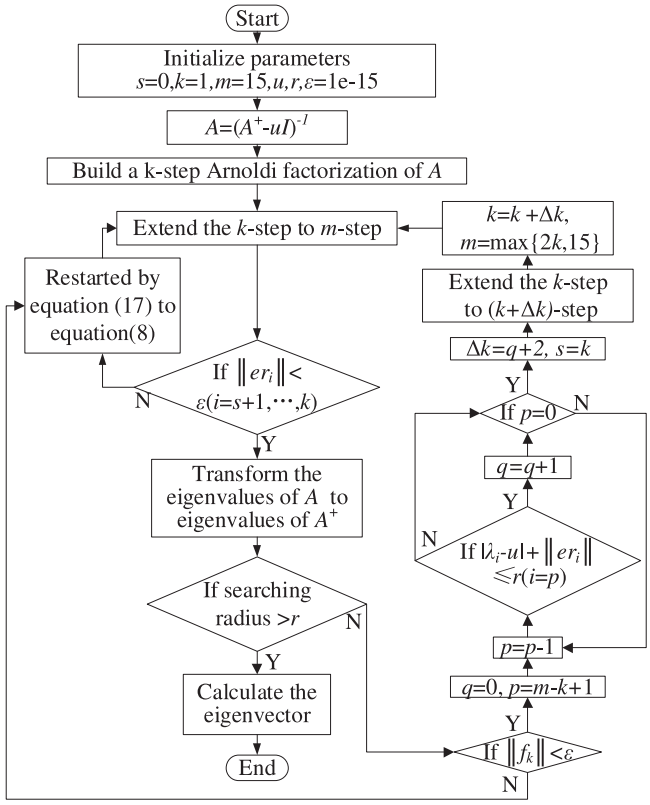

Fig. 1. Flowchart of the proposed algorithm.

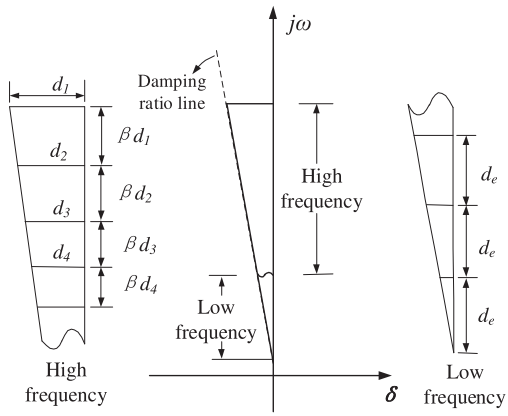

Fig. 2. Schema of the target region excision.

eigenvalues calculation in a specified circle (whose center and radius is u and r) is shown in Fig. 1.

The initial value of m is 15 because m is set as max{2k,15} in this proposed improved IRA algorithm.

To use the proposed improved IRA algorithm to compute all the eigenvalues in a given region, the target region is divided into small computing units and then using the searching circle, whose position and radius can be designed, to cover each computing unit. The proposed improved IRA algorithm has the following advantages: 1) the controllable parameters are transformed from the number of eigenvalues to the searching range with given center and radius; 2) the calculation process is automatically in searching all the critical eigenvalues in the concerned area without manual intervention; and 3) it avoids the missing of the desired eigenvalues.

F. Application of the Proposed Algorithm in Power System

Theoretically, the calculation method of eigenvalues in the specified circle and the dividing method of the target region are decoupled from each other. The target region is able to be reasonably divided according to its shape and size. After the division, the calculation of the eigenvalues of each computing unit is independent.

Aiming at searching all the poorly-damped LFO modes, this paper designs a division method for reference as shown in Fig. 2.

The target region concerned by LFO analysis is a narrow triangle region, in which the damping ratio ranges from 0 to 0.05 and the frequency ranges from 0.1 to 2.0 Hz. Firstly, the target region is divided into high frequency section and low frequency section and the high frequency section accounts for around 60% $\sim$ 80% of the total frequency period of the target region. The high frequency section is subdivided into numbers of right-angled trapezoid with a constant proportion of perpendicular and long base. The low frequency section is subdivided by 2 $\sim$ 5 right-angled trapezoids with the same altitude. If the low frequency section uses the same dividing method as the high frequency section, the number of computing units will be too much because the long bases of the right-angled trapezoids tend to zero.

G. Parametric Analysis

There are a few parameters and some relevant initial values should be given for the proposed method, such as β (the proportion of perpendicular and long base of the right-angled trapezoids in high frequency section), N_1 (the number of the right-angled trapezoids with the same altitude in low frequency section), ε, m and the initial value of k.

According to the division method mentioned before, the total number of computing units is determined by β and N_1. Since the calculation of all the computing units can be implemented in parallel, the parameters β and N_1 can be determined by the number of processors which is denoted by N_{par}. Assuming that the number of computing units in high frequency section is N_h, the frequency range of the target region is from f_0 to f_1 and the percentage of high frequency section is $\rho\%$, β and N_1 are constrained by (22) and (23).

$$d_1 + \sum_{k=2}^{N_h} \left(d_{k-1} - \frac{\beta d_{k-1}^2}{2\pi(f_1 - f_0)} \right) \le 2\pi(f_1 - f_0)\rho\% \quad (22)$$

$$N_h + N_l \le N_{par} \quad (23)$$

Considering the adaptability of the improved algorithm to different eigenvalues distribution situations, the initial value of k is strongly suggested as 1 to avoid unwanted computation. In

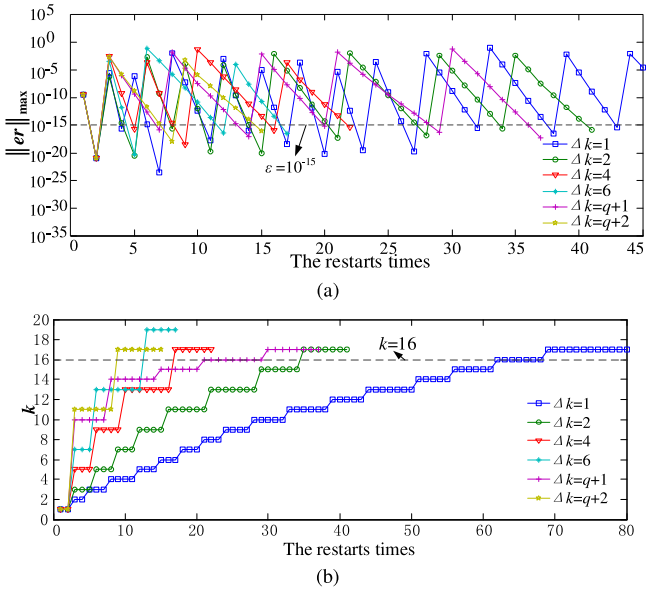

Fig. 3. Impacts on the algorithm performance of Δk: (a) Impact on $\|er\|$ of Δk and (b) Impact on k of Δk.

the restart calculation, the interesting information is compressed from the m-dimensional subspace into k-dimensional subspace, so $m > k$ must be satisfied.

IV. Numerical Experiments

A. Testing Environment

The numerical experiments in this paper are performed on the personal computer with a quad-core 2.9GHz processor, 4.0 GB RAM and Win10 operating system. The critical eigenvalues associated with LFO modes refer to those eigenvalues with the damping ratio ranges from 0 to 0.05 and the frequency ranges from 0.1 to 2.0 Hz. The target region involved in all experiments is shown in Fig. 2. The high frequency section accounts for 70% of the frequency range, $\beta = 1.2$ and $N_1 = 5$. $\varepsilon = 10^{-15}$. The initial value of k is 1 and $m = \max\{15, 2k\}$.

B. The impact of parameters on Algorithm Performance

In order to test the impact on algorithm performance of different values of Δk and m, the eigenvalues in a specific circle is calculated, which is performed on an actual power system. This system is Henan province power grid model, with 493 buses, 116 generators, 630 transmission lines, 226 transformers and 70 capacitors. There are 570 state variables for its LFO analysis. The center and the radius of the searching circle are respectively -0.15 + 9i and 0.4. There are actually 16 eigenvalues distributing in the searching circle.

1) The impact of Δk on algorithm performance

The converging processes are tested with six different values of Δk when $m = 2k$ shown in Fig. 3.

After each restart with expanding the dimension (namely increasing the value of k to $k + \Delta k$ and $m = 2k$), the value of $\|er\|$ will probably increase because new eigenvalues are not convergent. While after a few restarts without expanding the dimension (namely keeping the value of k and m constant), the value of $\|er\|$ will drop to less than ε. The total computation time is listed in Table I.

TABLE I
THE CALCULATION TIME UNDER VARIOUS VALUE OF ΔK WHEN M = 2K

Δk	1	2	4	6	$q+1$	$q+2$
$t(s)$	3.2700	2.1320	1.4980	1.3170	2.0740	1.2080

When $\Delta k = 1$, it needs many times to restart with expanding the dimension until the number of converged eigenvalues becomes more than 16. When $\Delta k = 6$, the number of converged eigenvalues rapidly surpasses 16. However, the number of eigenvalues acquired is actually 19 while 3 of them are outside the circle. When $\Delta k = q + 1$, the times of expanding the dimension increase while the number of converged eigenvalues is close to 16. The eigenvalues at the searching area boundary probably do not satisfy (21), which results in increment of restart times. However, $\Delta k = q + 2$ could avoid getting into this situation. When $\Delta k = q + 2$, its restarts times are apparently less than others.

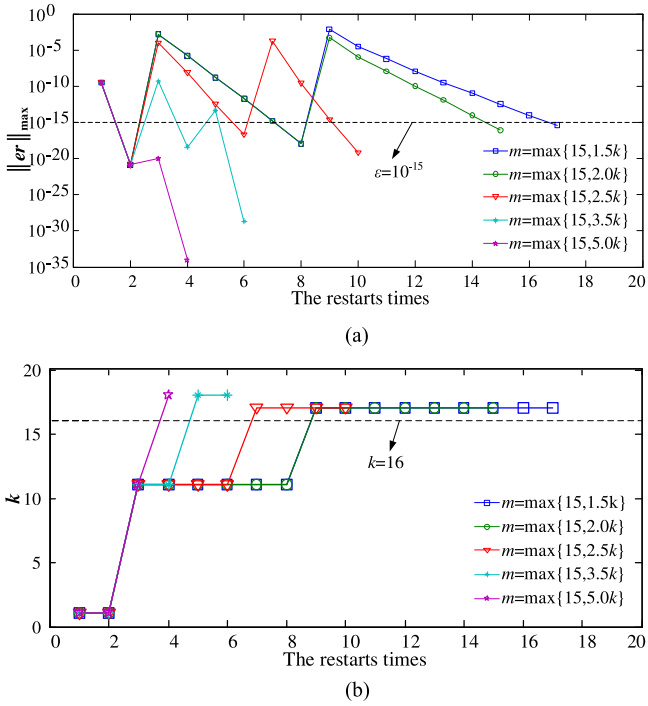

Fig. 4. Impacts on the algorithm performance of m: (a) Impact on $\|er\|$ of m and (b) Impact on k of m.

2) The impact of m on algorithm performance

The converging processes are tested with five different values of m ($\max\{15, 1.5k\}$, $\max\{15, 2.0k\}$, $\max\{15, 2.5k\}$, $\max\{15, 3.5k\}$ and $\max\{15, 5.0k\}$) when $\Delta k = q + 2$, as shown in Fig. 4.

When $m = \max\{15, 1.5k\}$, it tends to need many times to restart without expanding the dimension until the value of $\|er\|$

TABLE II
THE CALCULATION TIME UNDER VARIOUS VALUE OF M WHEN $\Delta k = q + 2$

m	max{15, 1.5k}	max{15, 2k}	max{15, 2.5k}	max{15, 3.5k}	max{15, 5k}
$t(s)$	1.4040	1.2080	1.1370	1.3100	1.4040

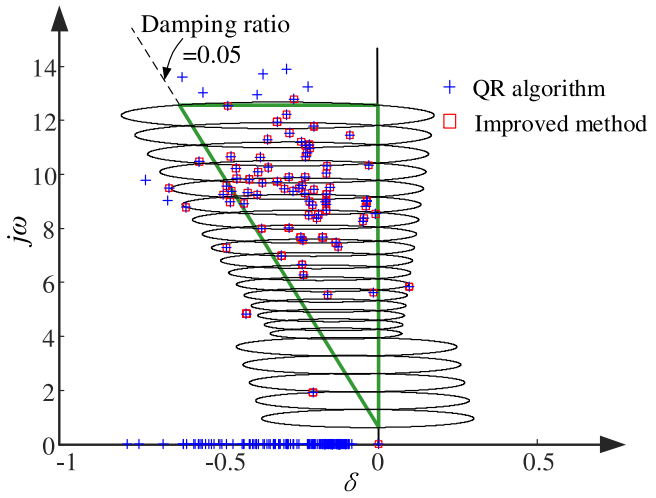

Fig. 5. Correctness verification of the proposed improved IRA algorithm. Here the triangle marks the target region and the circles are searching circles of the improved method.

drops to less than ε, although the time consumed by each restart is less. When m is large, each restart will converge more obviously and the number of restart without expanding the dimension will decrease which leads to more computation time. The total computation time is listed in Table II.

C. Correctness Verification

In order to verify the correctness of the proposed improved algorithm, the critical eigenvalues associated with LFO modes of the mentioned actual power system with 570 state variables are calculated by the proposed improved IRA algorithm and QR method.

The distribution of acquired eigenvalues and the corresponding searching circles are shown in Fig. 5.

The target region is covered by the searching circles and no eigenvalues are missed. Furthermore, the improved algorithm is able to complete the calculation process automatically without any manual intervention.

D. Effectiveness Verification

The Northeast power system of China, with 1873 buses, 2289 branches, 568 transformers, 332 generators and 1 back-to-back HVDC system, are performed using the proposed improved IRA method and the IRA method applied in PSD-SSAP. There are 5272 state variables for its LFO analysis. The method used in PSD-SSAP is named equidistant-searching IRA method in the rest of the paper. The equidistant-searching IRA method has two parameters: the number of shifts denoted by n_1 and the number of calculated eigenvalues for each shift point denoted by n_2. The line segment of the frequency range concerned on the imaginary axis of the complex plane is divided into n_1 segments on average. The midpoint of each segment is the center of a search circle. For each search circle, n_2 eigenvalues are computed by shift inverse transform. IRA(n_1, n_2) is denoted as the equidistant-searching IRA method under parameters of n_1 and n_2. IRA(improved) denotes the proposed IRA method. All the algorithms are implemented with Matlab language on sequential platform. The frequency range concerned is from 0.1 to 2.0 Hz.

TABLE III
THE COMPUTATION TIME OF VARIOUS SETS OF PARAMETERS FOR THE EQUIDISTANT-SEARCHING IRA METHOD

The number of shifts	The number of calculated eigenvalues for each shift point: 12	10	8	6	4
20	40.2040s/ 421.1273s	29.2487s/ 380.6535s	27.8789s/ 323.1066s	24.1485s/ 287.2173s	24.4541s/ 277.3172s
15	31.3156s/ 304.1477s	56.4175s/ 312.5274s	30.2125s/ 268.2155s	58.2298s/ 266.5053s	25.1847s*/ 178.6084s*
10	35.1583s/ 215.0335s	36.2854s/ 195.7308s	29.4912s/ 184.9178s	28.4993s/ 181.1828s	23.5031s*/ 128.2348s*

*missing a critical eigenvalues caused by inappropriate parameter settings.

1) Results of the equidistant-searching IRA method

Because the performance of the equidistant-searching IRA method is sensitive to parameters, two sets of parameters are adopted here: the empirical parameters and the more appropriate parameters after repeatedly adjusting.

Using the empirical parameters, the frequency range is searched with 20 shifts. For each shift point, 10 eigenvalues are calculated. There are only 13 concerned eigenvalues in the target region, while at least 200 eigenvalues are calculated. That is, by using the equidistant-searching IRA method with empirical parameters, most of the calculated eigenvalues are not necessary and cause seriously excessive computation. A manual adjustment for parameters is provided to find out more effective parameters in this method through testing many times repeatedly with various value of parameters. Because of the independent calculation characteristics of each computation unit, the computation time is determined by the search circle whose computation time is longest. If only the sequential calculation is considered, the computation time is determined by the sum of computation time of all the search circles. In Table III, the first rows for each parameter are the maximum computation time of single circle, and the second row is the sum of computation time. By comparing the results of various parameters, the best parameters for calculating simultaneously in parallel is IRA(20, 6) and the best parameters for sequential computation is IRA(10, 6).

The missing critical eigenvalue (-0.3767 + 11.6441i) by IRA (10, 4) and IRA (15, 4) is shown in Fig. 6, caused by too small parameter settings.

In the target region, the distribution density of the eigenvalues in the high frequency section and low frequency section are very

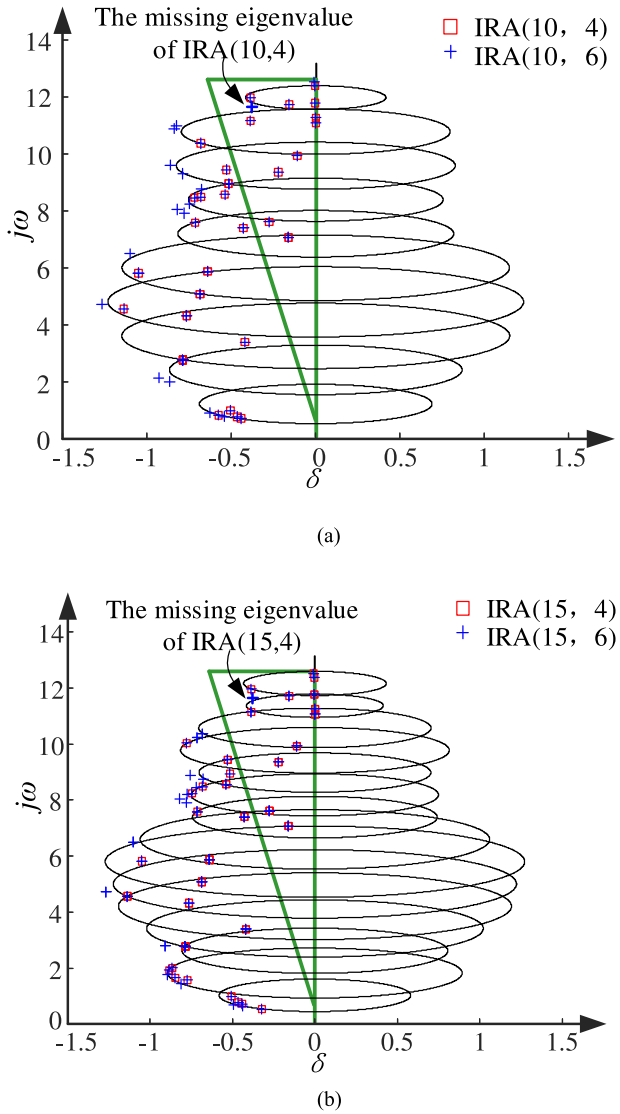

Fig. 6. The missing eigenvalues caused by improper parameter setting: (a) The missing eigenvalue of IRA(10, 4) and (b) The missing eigenvalue of IRA(15, 4). Here the triangle marks the target region and the circles in (a) and (b) are searching circles of the IRA (10, 4) and IRA (15, 4) respectively.

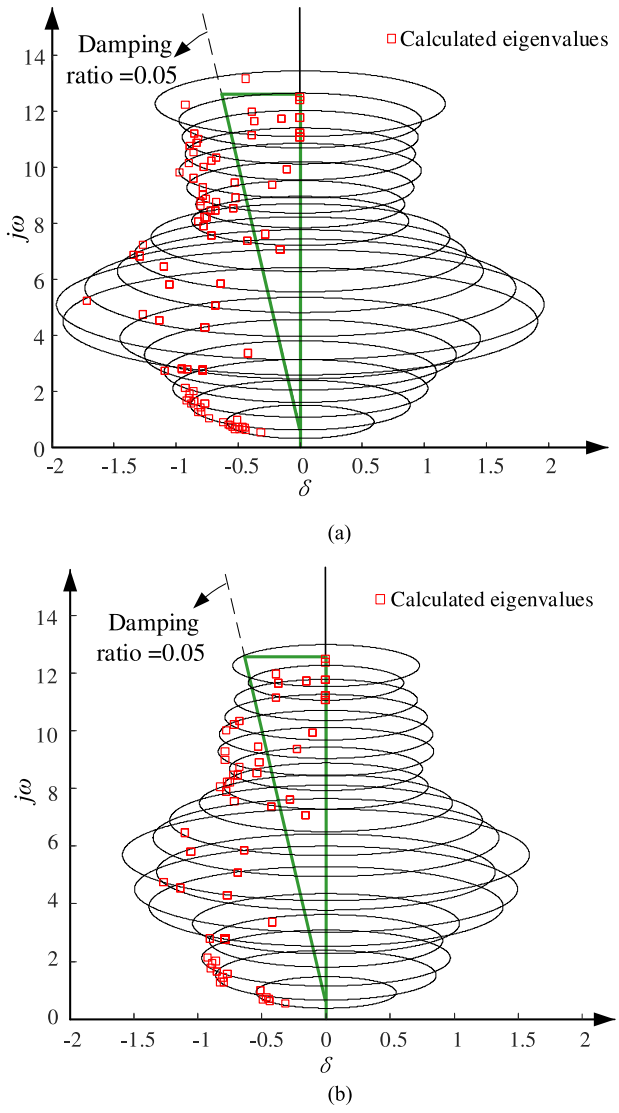

Fig. 7. Results of equidistant-searching IRA method: (a) IRA(20, 10) and (b) IRA(20, 6). Here the triangle marks the target region and the circles are searching circles of the improved method.

different. It is suitable for each shift point with 6 eigenvalues in high frequency section, but the calculation for each shift point with 6 eigenvalues in low frequency section are almost unnecessary, which can be seen in Fig. 7.

2) Results of the proposed automatic-expand IRA method

The result of the IRA(improved) is shown in Fig. 8, which illustrates that the proposed improved algorithm is able to automatically adjust the number of calculated eigenvalues according to the eigenvalues distribution.

There are only 46 calculated eigenvalues by utilizing the proposed improved method. The computation time is 16.4202 s by simultaneously computing the independent units, which is two third of computation time consumed by the equidistant-searching IRA method, as shown in Table III. Considering using sequential computation, the computation time is 178.5 s in total, which is almost the same as the minimum computation time consumed by the equidistant-searching IRA method.

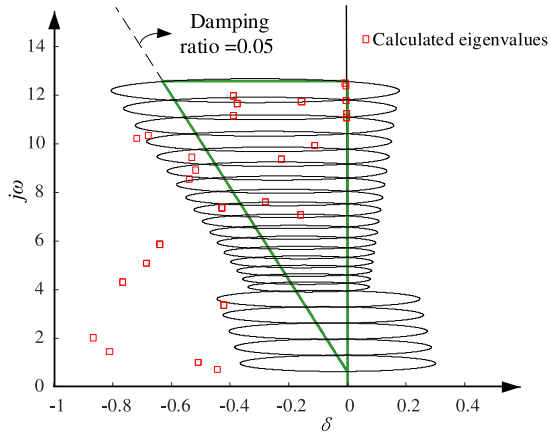

Fig. 8. Results of the proposed method. Here the triangle marks the target region and the circles are searching circles of the improved method.

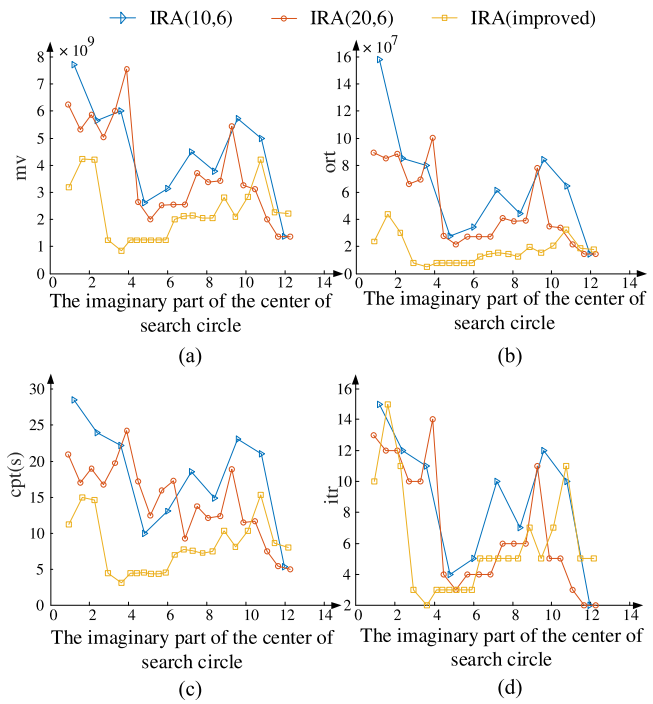

Fig 9. Detailed comparison in each search circle: (a) Number of matrix-vector products, (b) Number of orthogonalization products in matrix-vector products, (c) Computational time, and (d) Number of iterations.

TABLE IV
NUMERICAL RESULTS FOR SUM OF ALL SEARCH CIRCLES

	mv	*ort*	*itr*	*cpt*
IRA(improved)	4.780e10	3.477e8	124	178.5s
IRA(10,6)	4.544e10	6.532e8	88	183.5s
IRA(20,6)	7.527e10	9.452e8	136	287.2s

3) Detailed comparison of numerical results

Four detailed comparisons of three different calculation processes, the results of IRA(10, 6), IRA(20, 6) and IRA(improved) are shown in Fig. 9, including the number of the iterations (*itr*), the number of the matrix-vector products (*mv*), the computational time (*cpt*) in seconds and the number of orthogonalization products (*orts*) in matrix-vector products. The abscissa is the imaginary part of the search circle.

These results demonstrate that IRA (improved) is effective in reducing computation time for each search circle. For IRA(improved), the number of the matrix-vector products and the number of orthogonalization products in matrix-vector products both are smallest for most search circles. In Fig. 9, it can be seen that the computational time has the same trend as the number of the matrix-vector products.

Furthermore, the sum of the number of the matrix-vector products and the sum of the computational time of each circle in the improved algorithm is almost the same as those in the equidistant-searching IRA method with optimal parameters, shown in Table IV.

Although the number of search circles of IRA(improved) is more than twice of the number of search circles of IRA(10, 6), the numbers of matrix-vector products of IRA(improved) and IRA(10,6) have little difference. If only considering the sequential computation, the IRA(improved) also has the less computation time.

The numerical experiments show that the proposed improved algorithm can automatically finish the searching process without missing any eigenvalues and without manual intervention. The calculation efficiency is improved.

V. CONCLUSION

A Dynamic Augmented IRA algorithm is proposed in this paper for searching all the eigenvalues in the given region especially for LFO modes identification. Based on the locking mechanism, the controllable parameters are transformed from numbers to radius, which relieves the dependence of the computational performance on parameter setting and automatically fulfill the calculation process without missing any eigenvalues in the given region. Based on the principle of the proposed method, each computing unit is independent, which facilitates to utilize simultaneous computation in parallel for online calculation to improve the computational efficiency greatly. The main achievements of this paper are as follows.

1) The mechanism of expanding the dimension is analyzed, and it is proved that it will not lose the calculation efficiency after transform the controllable parameter from the number of eigenvalues to the radius of searching circle.
2) The implementation procedure of searching all the eigenvalues in a given region is provided and the eigenvalue searching problem in any kind of finite region including LFO analysis can be solved by the proposed algorithm.
3) Because of the advantages of the proposed algorithm, such as its automation, no eigenvalues missing caused by inappropriate parameter and the parallelizability, the improved algorithm is suitable for large scale system online analysis and calculation.

Two system with 570 order and 5272 order respective are used as test systems in this paper. The results show that the proposed improved algorithm is correct and effective.

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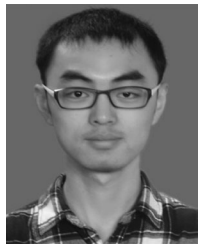

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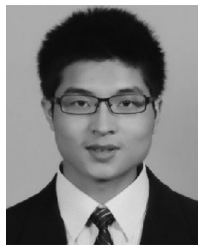

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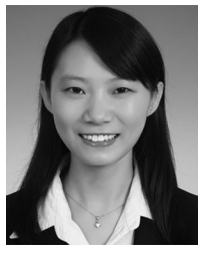

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